

THE DIMENSIONS OF THE HOMOMORPHISM SPACES TO INDECOMPOSABLE MODULES OVER THE FOUR SUBSPACE ALGEBRA

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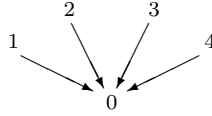
ANDRZEJ MRÓZ (TORUŃ, 2010)

Abstract. This is the addendum to the paper [3]. We give here the full proof of [3, Proposition 3.3], describing the formulas for the dimensions of the homomorphism spaces to indecomposable modules over the four subspace algebra Λ . The reader can also find here the full description of indecomposable Λ -modules.

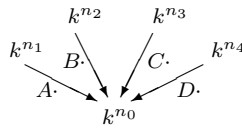
Introduction and preliminaries. We briefly recall the setting from [3]; for more details we refer to that article and also to [1, 2, 4, 5].

By k we denote the fixed algebraically closed field and by $\mathbb{M}_{n \times m}$, the set of all $(n \times m)$ -matrices over k , for $n, m \geq 0$ (we also consider the trivial matrices with zero rows or columns). Given $A \in \mathbb{M}_{m \times n}$, we denote by $r(A)$ the rank of A and by $\text{cor}(A) = m - r(A)$ its corank.

Let Q be the quiver



(the set of vertices $\{0, 1, 2, 3, 4\}$ we denote by Q_0). The path algebra $\Lambda = kQ$ we call the *four subspace algebra*. Then any collection of matrices $(A, B, C, D) \in \mathbb{M}_{n_0 \times n_1} \times \mathbb{M}_{n_0 \times n_2} \times \mathbb{M}_{n_0 \times n_3} \times \mathbb{M}_{n_0 \times n_4}$, $n_i \geq 0$, $i = 0, \dots, 4$, determines Λ -module (treated as the matrix representation of the quiver Q) of the form



Further on we will present Λ -modules as the quadruples (A, B, C, D) as above. For Λ -module M , the dimension vector $\underline{\dim}(M)$ we will present in the form

$$[\dim_k(M_0), \dim_k(M_1), \dim_k(M_2), \dim_k(M_3), \dim_k(M_4)].$$

By $\tau = \tau_\Lambda$ we denote the Auslander-Reiten translate in the category $\text{mod } \Lambda$ of finite-dimensional Λ -modules.

We recall that all the indecomposable Λ -modules are divided into the following three families:

- postprojective modules of the form $P(m, j) := \tau^{-m}P(j)$, for all $m \geq 0$, $j \in Q_0$, where $P(i)$ is the indecomposable projective Λ -module corresponding to the vertex i ;

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- preinjective modules of the form $I(m, j) := \tau^m I(j)$, for all $m \geq 0$, $j \in Q_0$, where $I(i)$ is the indecomposable injective Λ -module corresponding to the vertex i ;
- regular modules forming a family of pairwise orthogonal stable standard tubes \mathcal{T}_λ in Auslander-Reiten quiver of the category $\text{mod } \Lambda$, for all $\lambda \in \mathbb{P}^1(k)$, where $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_\infty$ are the tubes of rank 2 and the remaining are homogeneous. The modules from the tubes \mathcal{T}_λ , for $\lambda \in \{0, 1, \infty\}$, are denoted by $R(s, l, \lambda)$, where $s \in \mathbb{Z}_2$ is the number of quasi-socle, $l \geq 1$ is quasi-length (that is, $R(s, l, \lambda)$ has quasi-length l , its quasi-socle is isomorphic to $R(s, 1, \lambda)$, and moreover $\tau R(s, l, \lambda) \cong R(s \oplus_2 1, l, \lambda)$). The modules from homogeneous tubes \mathcal{T}_λ , for $\lambda \in k \setminus \{0, 1\}$, we denote by $R(l, \lambda)$, where $l \geq 1$ is quasi-length (in particular, $\tau R(l, \lambda) = R(l, \lambda)$).

The paper [3] deals with two natural problems: the *multiplicity problem* (the algorithm determining the multiplicities of all the indecomposables appearing in the indecomposable decomposition of Λ -module) and the *isomorphism problem* (the algorithmic criterion for deciding if two Λ -modules are isomorphic).

In the above considerations the crucial role is played by the direct formulas for computing the dimensions $[M, X] := \dim_k(\text{Hom}_\Lambda(M, X))$ of homomorphism spaces from given Λ -module M to arbitrary indecomposable Λ -module X . Such formulas were given in [3, Proposition 3.3]. The aim of this paper is to give the full proof of the proposition, which we recall below in 1.

In the proof, given in 3, we use the matrix description of the indecomposable Λ -modules, which we recall in Proposition 2.

1. We recall below the assertion of [3, Proposition 3.3]. First we establish some notation. For the integers $x, y, x', y', u, v, s, t, i \geq 0$ with $u \leq x$, $v \leq y, y'$, $s \leq x, x'$, $t \leq y$ and the matrices $Z \in \mathbb{M}_{x \times y}$, $Z' \in \mathbb{M}_{x' \times y'}$, $F \in \mathbb{M}_{u \times v}$, $E \in \mathbb{M}_{s \times t}$ we define the block matrices $\mathcal{M}_1^i = \mathcal{M}_1(Z, Z', F, i)$, $\mathcal{M}_2^i = \mathcal{M}_2(Z, Z', E, i) \in \mathbb{M}_{(x'+ix) \times (y'+iy)}$ and $\mathcal{M}_3^i = \mathcal{M}_3(Z, Z', E, i) \in \mathbb{M}_{(x'+ix) \times (y'+(i+1)y)}$ as follows:

$$\mathcal{M}_1^i := \begin{bmatrix} Z' & & & & \\ F'' & Z & & & \\ & F' & Z & & \\ & & \ddots & \ddots & \\ & & & F' & Z \end{bmatrix}, \quad \mathcal{M}_2^i := \begin{bmatrix} Z' & E'' & & & \\ & Z & E' & & \\ & & \ddots & \ddots & \\ & & & Z & E' \\ & & & & Z \end{bmatrix},$$

$$\mathcal{M}_3^i := \begin{bmatrix} Z' & E'' & & & & \\ & Z & E' & & & \\ & & \ddots & \ddots & & \\ & & & Z & E' & \\ & & & & Z & E' \end{bmatrix}$$

(the blocks Z' , E'' , F'' appear only once, the remaining coefficients are zero), where $F' = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, $E' = \begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix} \in \mathbb{M}_{x \times y}$, $F'' = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_{x \times y'}$ and $E'' = \begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix} \in \mathbb{M}_{x' \times y}$. In particular, $\mathcal{M}_1^0 = \mathcal{M}_2^0 = Z'$ and $\mathcal{M}_3^0 = [Z' | E'']$.

For $A \in \mathbb{M}_{x \times y}$, $B \in \mathbb{M}_{z \times t}$, by $A \oplus B$ we mean the matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{M}_{(x+z) \times (y+t)}$.

Proposition. Let $M = (A, B, C, D)$ be a fixed Λ -module, with $A \in \mathbb{M}_{n_0 \times n_1}$, $B \in \mathbb{M}_{n_0 \times n_2}$, $C \in \mathbb{M}_{n_0 \times n_3}$, $D \in \mathbb{M}_{n_0 \times n_4}$. Then the formulae for the dimension $[M, X]$, for indecomposable Λ -module $X \in \mathcal{X}$, are given in the following table

X	$[M, X]$
$P(0, 0)$	$\text{cor}([A \ B \ C \ D])$
$P(n, 0),$ $n \geq 1$	$\text{cor}(\mathcal{M}_3(Z_{P(n,0)}, Z'_{P(n,0)}, C \oplus D, n-1),$ $Z'_{P(n,0)} = \begin{bmatrix} A & 0 & B & 0 & C & D \\ 0 & 0 & 0 & B & 0 & -D \\ 0 & A & 0 & 0 & -C & 0 \end{bmatrix}, Z_{P(n,0)} = \begin{bmatrix} 0 & -D & 0 & B \\ -C & 0 & A & 0 \end{bmatrix}$
$P(2n+1, 1),$ $n \geq 0$	$\text{cor}(\mathcal{M}_3(Z_{P(2n+1,1)}, Z_{P(2n+1,1)}, -A, n)),$ $Z_{P(2n+1,1)} = \begin{bmatrix} A & 0 & C & D \\ 0 & B & 0 & -D \end{bmatrix}$
$P(2n+1, 2),$ $n \geq 0$	$\text{cor}(\mathcal{M}_3(Z_{P(2n+1,2)}, Z_{P(2n+1,2)}, -B, n)),$ $Z_{P(2n+1,2)} = \begin{bmatrix} B & 0 & D & A \\ 0 & C & 0 & -A \end{bmatrix}$
$P(2n+1, 3),$ $n \geq 0$	$\text{cor}(\mathcal{M}_3(Z_{P(2n+1,3)}, Z_{P(2n+1,3)}, -C, n)),$ $Z_{P(2n+1,3)} = \begin{bmatrix} C & 0 & A & B \\ 0 & D & 0 & -B \end{bmatrix}$
$P(2n+1, 4),$ $n \geq 0$	$\text{cor}(\mathcal{M}_3(Z_{P(2n+1,4)}, Z_{P(2n+1,4)}, -D, n)),$ $Z_{P(2n+1,4)} = \begin{bmatrix} D & 0 & B & C \\ 0 & A & 0 & -C \end{bmatrix}$
$P(2n, 1),$ $n \geq 0$	$\text{cor}(\mathcal{M}_1(Z_{P(2n,1)}, Z'_{P(2n,1)}, -D, n),$ $Z'_{P(2n,1)} = [\ B \ C \ D \], Z_{P(2n,1)} = \begin{bmatrix} C & A & 0 & 0 \\ -C & 0 & B & D \end{bmatrix}$
$P(2n, 2),$ $n \geq 0$	$\text{cor}(\mathcal{M}_1(Z_{P(2n,2)}, Z'_{P(2n,2)}, -A, n),$ $Z'_{P(2n,2)} = [\ C \ D \ A \], Z_{P(2n,2)} = \begin{bmatrix} D & B & 0 & 0 \\ -D & 0 & C & A \end{bmatrix}$
$P(2n, 3),$ $n \geq 0$	$\text{cor}(\mathcal{M}_1(Z_{P(2n,3)}, Z'_{P(2n,3)}, -B, n),$ $Z'_{P(2n,3)} = [\ D \ A \ B \], Z_{P(2n,3)} = \begin{bmatrix} A & C & 0 & 0 \\ -A & 0 & D & B \end{bmatrix}$
$P(2n, 4),$ $n \geq 0$	$\text{cor}(\mathcal{M}_1(Z_{P(2n,4)}, Z'_{P(2n,4)}, -C, n),$ $Z'_{P(2n,4)} = [\ A \ B \ C \], Z_{P(2n,4)} = \begin{bmatrix} B & D & 0 & 0 \\ -B & 0 & A & C \end{bmatrix}$
$I(0, 0)$	n_0
$I(n, 0),$ $n \geq 1$	$\text{cor}(\mathcal{M}_2(Z_{I(n,0)}, Z'_{I(n,0)}, D \oplus C, n-1)),$ $Z'_{I(n,0)} = \begin{bmatrix} D & C & 0 & 0 \\ 0 & -C & 0 & B \\ -D & 0 & A & 0 \end{bmatrix}, Z_{I(n,0)} = \begin{bmatrix} 0 & -C & 0 & B \\ -D & 0 & A & 0 \end{bmatrix}$
$I(2n+1, 1),$ $n \geq 0$	$\text{cor}(\mathcal{M}_2(Z_{I(2n+1,1)}, A, -C, n)),$ $Z_{I(2n+1,1)} = \begin{bmatrix} C & D & 0 & B \\ 0 & -D & A & 0 \end{bmatrix}$
$I(2n+1, 2),$ $n \geq 0$	$\text{cor}(\mathcal{M}_2(Z_{I(2n+1,2)}, B, -D, n)),$ $Z_{I(2n+1,2)} = \begin{bmatrix} D & A & 0 & C \\ 0 & -A & B & 0 \end{bmatrix}$
$I(2n+1, 3),$ $n \geq 0$	$\text{cor}(\mathcal{M}_2(Z_{I(2n+1,3)}, C, -A, n)),$ $Z_{I(2n+1,3)} = \begin{bmatrix} A & B & 0 & D \\ 0 & -B & C & 0 \end{bmatrix}$
$I(2n+1, 4),$ $n \geq 0$	$\text{cor}(\mathcal{M}_2(Z_{I(2n+1,4)}, D, -B, n)),$ $Z_{I(2n+1,4)} = \begin{bmatrix} B & C & 0 & A \\ 0 & -C & D & 0 \end{bmatrix}$

$I(0, 1)$	n_1
$I(2n, 1),$	$\text{cor}(\mathcal{M}_2(Z_{I(2n,1)}, Z'_{I(2n,1)}, -A, n-1)),$
$n \geq 1$	$Z'_{I(2n,1)} = \begin{bmatrix} B & 0 & D \\ 0 & C & -D \end{bmatrix}, Z_{I(2n,1)} = \begin{bmatrix} A & B & 0 & D \\ 0 & 0 & C & -D \end{bmatrix}$
$I(0, 2)$	n_2
$I(2n, 2),$	$\text{cor}(\mathcal{M}_2(Z_{I(2n,2)}, Z'_{I(2n,2)}, -B, n-1)),$
$n \geq 1$	$Z'_{I(2n,2)} = \begin{bmatrix} C & 0 & A \\ 0 & D & -A \end{bmatrix}, Z_{I(2n,2)} = \begin{bmatrix} B & C & 0 & A \\ 0 & 0 & D & -A \end{bmatrix}$
$I(0, 3)$	n_3
$I(2n, 3),$	$\text{cor}(\mathcal{M}_2(Z_{I(2n,3)}, Z'_{I(2n,3)}, -C, n-1)),$
$n \geq 1$	$Z'_{I(2n,3)} = \begin{bmatrix} D & 0 & B \\ 0 & A & -B \end{bmatrix}, Z_{I(2n,3)} = \begin{bmatrix} C & D & 0 & B \\ 0 & 0 & A & -B \end{bmatrix}$
$I(0, 4)$	n_4
$I(2n, 4),$	$\text{cor}(\mathcal{M}_2(Z_{I(2n,4)}, Z'_{I(2n,4)}, -D, n-1)),$
$n \geq 1$	$Z'_{I(2n,4)} = \begin{bmatrix} A & 0 & C \\ 0 & B & -C \end{bmatrix}, Z_{I(2n,4)} = \begin{bmatrix} D & A & 0 & C \\ 0 & 0 & B & -C \end{bmatrix}$
$R(l, \lambda),$	$\text{cor}(\mathcal{M}_2(Z_{R(l,\lambda)}, Z_{R(l,\lambda)}, -D, l-1)),$
$l \geq 1, \lambda \neq 0, 1$	$Z_{R(l,\lambda)} = \begin{bmatrix} D & C & B & 0 \\ -\lambda D & -C & 0 & A \end{bmatrix}$
$R(0, 2l, 0),$	$\text{cor}(\mathcal{M}_2(Z_{R(0,2l,0)}, Z_{R(0,2l,0)}, -D, l-1)),$
$l \geq 1$	$Z_{R(0,2l,0)} = \begin{bmatrix} D & C & B & 0 \\ 0 & -C & 0 & A \end{bmatrix}$
$R(1, 2l, 0),$	$\text{cor}(\mathcal{M}_2(Z_{R(1,2l,0)}, Z_{R(1,2l,0)}, -C, l-1)),$
$l \geq 1$	$Z_{R(1,2l,0)} = \begin{bmatrix} C & D & A & 0 \\ 0 & -D & 0 & B \end{bmatrix}$
$R(0, 2l, 1),$	$\text{cor}(\mathcal{M}_2(Z_{R(0,2l,1)}, Z_{R(0,2l,1)}, -B, l-1)),$
$l \geq 1$	$Z_{R(0,2l,1)} = \begin{bmatrix} B & D & A & 0 \\ 0 & -D & 0 & C \end{bmatrix}$
$R(1, 2l, 1),$	$\text{cor}(\mathcal{M}_2(Z_{R(1,2l,1)}, Z_{R(1,2l,1)}, -D, l-1)),$
$l \geq 1$	$Z_{R(1,2l,1)} = \begin{bmatrix} D & B & C & 0 \\ 0 & -B & 0 & A \end{bmatrix}$
$R(0, 2l, \infty),$	$\text{cor}(\mathcal{M}_2(Z_{R(0,2l,\infty)}, Z_{R(0,2l,\infty)}, -D, l-1)),$
$l \geq 1$	$Z_{R(0,2l,\infty)} = \begin{bmatrix} D & C & A & 0 \\ 0 & -C & 0 & B \end{bmatrix}$
$R(1, 2l, \infty),$	$\text{cor}(\mathcal{M}_2(Z_{R(1,2l,\infty)}, Z_{R(1,2l,\infty)}, -C, l-1)),$
$l \geq 1$	$Z_{R(1,2l,\infty)} = \begin{bmatrix} C & D & B & 0 \\ 0 & -D & 0 & A \end{bmatrix}$
$R(0, 2l-1, 0),$	$\text{cor}(\mathcal{M}_2(Z_{R(0,2l-1,0)}, Z'_{R(0,2l-1,0)}, -B, l-1)),$
$l \geq 1$	$Z'_{R(0,2l-1,0)} = \begin{bmatrix} A & C \end{bmatrix}, Z_{R(0,2l-1,0)} = \begin{bmatrix} B & D & 0 & A \\ 0 & 0 & C & -A \end{bmatrix}$
$R(1, 2l-1, 0),$	$\text{cor}(\mathcal{M}_2(Z_{R(1,2l-1,0)}, Z'_{R(1,2l-1,0)}, -A, l-1)),$
$l \geq 1$	$Z'_{R(1,2l-1,0)} = \begin{bmatrix} B & D \end{bmatrix}, Z_{R(1,2l-1,0)} = \begin{bmatrix} A & C & 0 & B \\ 0 & 0 & D & -B \end{bmatrix}$
$R(0, 2l-1, 1),$	$\text{cor}(\mathcal{M}_2(Z_{R(0,2l-1,1)}, Z'_{R(0,2l-1,1)}, -A, l-1)),$
$l \geq 1$	$Z'_{R(0,2l-1,1)} = \begin{bmatrix} C & D \end{bmatrix}, Z_{R(0,2l-1,1)} = \begin{bmatrix} A & B & 0 & C \\ 0 & 0 & D & -C \end{bmatrix}$

$R(1, 2l-1, 1),$	$\text{cor}(\mathcal{M}_2(Z_{R(1, 2l-1, 1)}, Z'_{R(1, 2l-1, 1)}, -C, l-1)),$
$l \geq 1$	$Z'_{R(1, 2l-1, 1)} = \begin{bmatrix} A & B \end{bmatrix}, Z_{R(1, 2l-1, 1)} = \begin{bmatrix} C & D & 0 & A \\ 0 & 0 & B & -A \end{bmatrix}$
$R(0, 2l-1, \infty),$	$\text{cor}(\mathcal{M}_2(Z_{R(0, 2l-1, \infty)}, Z'_{R(0, 2l-1, \infty)}, -A, l-1)),$
$l \geq 1$	$Z'_{R(0, 2l-1, \infty)} = \begin{bmatrix} B & C \end{bmatrix}, Z_{R(0, 2l-1, \infty)} = \begin{bmatrix} A & D & 0 & B \\ 0 & 0 & C & -B \end{bmatrix}$
$R(1, 2l-1, \infty),$	$\text{cor}(\mathcal{M}_2(Z_{R(1, 2l-1, \infty)}, Z'_{R(1, 2l-1, \infty)}, -B, l-1)),$
$l \geq 1$	$Z'_{R(1, 2l-1, \infty)} = \begin{bmatrix} A & D \end{bmatrix}, Z_{R(1, 2l-1, \infty)} = \begin{bmatrix} B & C & 0 & A \\ 0 & 0 & D & -A \end{bmatrix}$

2. In the proof of Proposition 1 we essentially use the matrix description of all the indecomposable modules over the four subspace algebra Λ , which we recall below (see [4, 5]; note that in the description of indecomposables from the cited handbooks one can find some number of misprints and little errors, which sometimes caused that presented modules were decomposable; we corrected the description here).

For $m, n \geq 0$, by $0_{m \times n} \in \mathbb{M}_{m \times n}$ or $0_n \in \mathbb{M}_{n \times n}$ we denote the zero matrix and by $I_n \in \mathbb{M}_{n \times n}$, the identity matrix. Moreover we set

$$\begin{aligned} {}^\circ\pi_{n, n+1} &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & 0 \end{bmatrix} \in \mathbb{M}_{n \times (n+1)}, \quad \pi_{n, n+1}^\circ = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathbb{M}_{n \times (n+1)}, \\ \bar{I}_n &= \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{bmatrix} \in \mathbb{M}_{n \times n}, \quad J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \in \mathbb{M}_{n \times n}, \end{aligned}$$

for $n \geq 0$, $\lambda \in k$, where the remaining coefficients are zero (note that $J_n(\lambda)$ is the upper-triangular Jordan block with eigenvalue λ).

Proposition. *Let Λ be the four subspace algebra. Then the fixed matrix representatives of all the isomorphism classes of indecomposable Λ -modules and their dimension vectors are given in the following table:*

$P(n, 0),$	$[2n+1, n, n, n, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_n \\ 0_{n+1, n} \end{bmatrix}, \begin{bmatrix} 0_{n+1, n} \\ I_n \end{bmatrix}, \begin{bmatrix} 0_{1, n} \\ I_n \\ \bar{I}_n \end{bmatrix}, \begin{bmatrix} I_n \\ \bar{I}_n \\ 0_{1, n} \end{bmatrix} \right)$
$P(2n+1, 1),$	$[2n+2, n, n+1, n+1, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} 0_{1, n} \\ I_n \\ I_n \\ 0_{1, n} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ I_{n+1} \end{bmatrix} \right)$
$P(2n+1, 2),$	$[2n+2, n+1, n, n+1, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} I_{n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} 0_{1, n} \\ I_n \\ I_n \\ 0_{1, n} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1} \\ I_{n+1} \end{bmatrix} \right)$
$P(2n+1, 3),$	$[2n+2, n+1, n+1, n, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} 0_{n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} 0_{1, n} \\ I_n \\ I_n \\ 0_{1, n} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n+1} \end{bmatrix} \right)$

$P(2n+1, 4),$	$[2n+2, n+1, n+1, n+1, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_{n+1} \\ 0_{n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} 0_{1,n} \\ I_n \\ I_n \\ 0_{1,n} \end{bmatrix} \right)$
$P(2n, 1),$	$[2n+1, n+1, n, n, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} 0_{1,n} \\ I_n \\ I_n \\ 0_{1,n} \end{bmatrix}, \begin{bmatrix} I_n \\ 0_{1,n} \\ I_n \end{bmatrix} \right)$
$P(2n, 2),$	$[2n+1, n, n+1, n, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_n \\ 0_{1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} 0_{1,n} \\ I_n \\ I_n \end{bmatrix} \right)$
$P(2n, 3),$	$[2n+1, n, n, n+1, n],$
$n \geq 0$	$\left(\begin{bmatrix} 0_{1,n} \\ I_n \\ I_n \end{bmatrix}, \begin{bmatrix} I_n \\ 0_{1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix} \right)$
$P(2n, 4),$	$[2n+1, n, n, n, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} 0_{1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_n \\ 0_{1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix} \right)$
$I(n, 0),$	$[2n+1, n+1, n+1, n+1, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} 0_{n,n+1} \\ I_{n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} \bar{I}_{n+1} \\ \circ\pi_{n,n+1} \end{bmatrix}, \begin{bmatrix} \pi_{n,n+1}^\circ \\ \bar{I}_{n+1} \end{bmatrix} \right)$
$I(2n+1, 1),$	$[2n+1, n, n+1, n+1, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ \circ\pi_{n,n+1} \end{bmatrix} \right)$
$I(2n+1, 2),$	$[2n+1, n+1, n, n+1, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} I_{n+1} \\ \circ\pi_{n,n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix} \right)$
$I(2n+1, 3),$	$[2n+1, n+1, n+1, n, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} I_{n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ \circ\pi_{n,n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix} \right)$
$I(2n+1, 4),$	$[2n+1, n+1, n+1, n+1, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix}, \begin{bmatrix} I_{n+1} \\ \circ\pi_{n,n+1} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix} \right)$
$I(2n, 1),$	$[2n, n+1, n, n, n],$
$n \geq 0$	$\left(\begin{bmatrix} \circ\pi_{n,n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix}, \begin{bmatrix} 0_n \\ I_n \end{bmatrix}, \begin{bmatrix} I_n \\ 0_n \end{bmatrix}, \begin{bmatrix} I_n \\ I_n \end{bmatrix} \right)$
$I(2n, 2),$	$[2n, n, n+1, n, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_n \\ I_n \end{bmatrix}, \begin{bmatrix} \circ\pi_{n,n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix}, \begin{bmatrix} 0_n \\ I_n \end{bmatrix}, \begin{bmatrix} I_n \\ 0_n \end{bmatrix} \right)$
$I(2n, 3),$	$[2n, n, n, n+1, n],$
$n \geq 0$	$\left(\begin{bmatrix} I_n \\ 0_n \end{bmatrix}, \begin{bmatrix} I_n \\ I_n \end{bmatrix}, \begin{bmatrix} \circ\pi_{n,n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix}, \begin{bmatrix} 0_n \\ I_n \end{bmatrix} \right)$
$I(2n, 4),$	$[2n, n, n, n, n+1],$
$n \geq 0$	$\left(\begin{bmatrix} 0_n \\ I_n \end{bmatrix}, \begin{bmatrix} I_n \\ 0_n \end{bmatrix}, \begin{bmatrix} I_n \\ I_n \end{bmatrix}, \begin{bmatrix} \circ\pi_{n,n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix} \right)$

$R(l, \lambda),$ $l \geq 1, \lambda \in k \setminus \{0, 1\}$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix}, \begin{bmatrix} J_l(\lambda) \\ I_l \end{bmatrix} \right)$
$R(0, 2l, 0),$ $l \geq 1$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix}, \begin{bmatrix} J_l(0) \\ I_l \end{bmatrix} \right)$
$R(1, 2l, 0),$ $l \geq 1$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} J_l(0) \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix} \right)$
$R(0, 2l, 1),$ $l \geq 1$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} J_l(0) \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix} \right)$
$R(1, 2l, 1),$ $l \geq 1$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix}, \begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} J_l(0) \\ I_l \end{bmatrix} \right)$
$R(0, 2l, \infty),$ $l \geq 1$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix}, \begin{bmatrix} J_l(0) \\ I_l \end{bmatrix} \right)$
$R(1, 2l, \infty),$ $l \geq 1$	$[2l, l, l, l, l],$ $\left(\begin{bmatrix} I_l \\ 0_l \end{bmatrix}, \begin{bmatrix} 0_l \\ I_l \end{bmatrix}, \begin{bmatrix} J_l(0) \\ I_l \end{bmatrix}, \begin{bmatrix} I_l \\ I_l \end{bmatrix} \right)$
$R(0, 2l-1, 0),$ $l \geq 1$	$[2l-1, l-1, l, l-1, l],$ $\left(\begin{bmatrix} I_{l-1} \\ 0_{1, l-1} \\ I_{l-1} \end{bmatrix}, \begin{bmatrix} \circ \pi_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{l, l-1} \end{bmatrix}, \begin{bmatrix} 0_{l-1, l} \\ I_l \end{bmatrix} \right)$
$R(1, 2l-1, 0),$ $l \geq 1$	$[2l-1, l, l-1, l, l-1],$ $\left(\begin{bmatrix} \circ \pi_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{1, l-1} \\ I_{l-1} \end{bmatrix}, \begin{bmatrix} 0_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{l, l-1} \end{bmatrix} \right)$
$R(0, 2l-1, 1),$ $l \geq 1$	$[2l-1, l, l, l-1, l-1],$ $\left(\begin{bmatrix} \circ \pi_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} 0_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{1, l-1} \\ I_{l-1} \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{l, l-1} \end{bmatrix} \right)$
$R(1, 2l-1, 1),$ $l \geq 1$	$[2l-1, l-1, l-1, l, l],$ $\left(\begin{bmatrix} I_{l-1} \\ 0_{1, l-1} \\ I_{l-1} \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{l, l-1} \end{bmatrix}, \begin{bmatrix} \circ \pi_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} 0_{l-1, l} \\ I_l \end{bmatrix} \right)$
$R(0, 2l-1, \infty),$ $l \geq 1$	$[2l-1, l, l-1, l-1, l],$ $\left(\begin{bmatrix} \circ \pi_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{1, l-1} \\ I_{l-1} \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{l, l-1} \end{bmatrix}, \begin{bmatrix} 0_{l-1, l} \\ I_l \end{bmatrix} \right)$
$R(1, 2l-1, \infty),$ $l \geq 1$	$[2l-1, l-1, l, l, l-1],$ $\left(\begin{bmatrix} I_{l-1} \\ 0_{1, l-1} \\ I_{l-1} \end{bmatrix}, \begin{bmatrix} \circ \pi_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} 0_{l-1, l} \\ I_l \end{bmatrix}, \begin{bmatrix} I_{l-1} \\ 0_{l, l-1} \end{bmatrix} \right)$

One proves the proposition using the standard arguments as in [5, 4].

Remark. Note that for any $l \geq 1$ we have $R(0, 2l, 0) = R(l, \lambda)$ if we substitute

$\lambda := 0$. One can also easily check that $R(1, 2l, 1)$ is isomorphic to $R(l, \lambda)$ with $\lambda := 1$. Therefore for every $\lambda \in k \setminus \{0, 1\}$, the tube \mathcal{T}_λ contains the indecomposable module $R(l, \lambda)$, for any $l \geq 1$.

3. Proof of Proposition 1. Let $M = (A, B, C, D)$ be a fixed Λ -module, with $A \in \mathbb{M}_{n_0 \times n_1}$, $B \in \mathbb{M}_{n_0 \times n_2}$, $C \in \mathbb{M}_{n_0 \times n_3}$, $D \in \mathbb{M}_{n_0 \times n_4}$. Recall [1] that for Λ -module $X = (A', B', C', D')$ with $A' \in \mathbb{M}_{m_0 \times m_1}$, $B' \in \mathbb{M}_{m_0 \times m_2}$, $C' \in \mathbb{M}_{m_0 \times m_3}$, $D' \in \mathbb{M}_{m_0 \times m_4}$, any Λ -homomorphism $F : M \rightarrow X$ is the collection $(F_0, F_1, F_2, F_3, F_4)$ of k -homomorphisms $F_i : k^{n_i} \rightarrow k^{m_i}$, $i \in Q_0$, satisfying the commutativity relations

$$(*) \quad F_0 A = A' F_1, \quad F_0 B = B' F_2, \quad F_0 C = C' F_3, \quad F_0 D = D' F_4$$

(we identify F_i with its matrix from $\mathbb{M}_{m_i \times n_i}$ given in the standard bases). So the dimension $[M, X]$ equals the dimension of the solution space of the system $(*)$ of linear equations with the matrices F_i treated as variables. We calculate below these dimensions for every indecomposable Λ -module X . We use the description of indecomposables from Proposition 2.

Case 1: $X = P(n, 0)$. $P(0, 0)$ is the projective Λ -module $P(0) = (\varepsilon_{1,0}, \varepsilon_{1,0}, \varepsilon_{1,0}, \varepsilon_{1,0})$. So every homomorphism from M to $P(0, 0)$ is given by the collection $(F_0, F_1, F_2, F_3, F_4) = (y, 0, 0, 0, 0)$, $y \in \mathbb{M}_{1 \times n_0}$, such that $yA = 0$, $yB = 0$, $yC = 0$, $yD = 0$ that is, satisfying the matrix equation $y[A \ B \ C \ D] = 0$. Thus $[M, P(0, 0)] = \text{cor}([A \ B \ C \ D])$.

Now fix $n \geq 1$. Recall that $P(n, 0) = \left(\begin{bmatrix} I_n \\ 0_{n+1,n} \end{bmatrix}, \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix}, \begin{bmatrix} 0_{1,n} \\ I_n \\ \bar{I}_n \end{bmatrix}, \begin{bmatrix} I_n \\ \bar{I}_n \\ 0_{1,n} \end{bmatrix} \right)$ (see Proposition 2). So every homomorphism from M to $P(n, 0)$ is given by the collection of matrices $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{(2n+1) \times n_0} \times \mathbb{M}_{n \times n_1} \times \mathbb{M}_{n \times n_2} \times \mathbb{M}_{n \times n_3} \times \mathbb{M}_{n \times n_4}$ satisfying the commutativity relations

$$(*)^1 \quad F_0 A = \begin{bmatrix} I_n \\ 0_{n+1,n} \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} 0_{1,n} \\ I_n \\ \bar{I}_n \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} I_n \\ \bar{I}_n \\ 0_{1,n} \end{bmatrix} F_4.$$

Let us denote by y_1, \dots, y_{2n+1} , s_1, \dots, s_n , t_1, \dots, t_n , u_1, \dots, u_n and respectively w_1, \dots, w_n , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . So the system $(*)^1$ can be presented in the form

$$(**)^1 \quad \begin{cases} y_1 A = s_1 \\ \vdots \\ y_n A = s_n \\ y_{n+1} A = 0 \\ \vdots \\ y_{2n} A = 0 \\ y_{2n+1} A = 0 \end{cases} \begin{cases} y_1 B = 0 \\ \vdots \\ y_n B = 0 \\ y_{n+1} B = 0 \\ y_{n+2} B = t_1 \\ \vdots \\ y_{2n+1} B = t_n \end{cases} \begin{cases} y_1 C = 0 \\ y_2 C = u_1 \\ \vdots \\ y_{n+1} C = u_n \\ y_{n+2} C = u_n \\ \vdots \\ y_{2n+1} C = u_1 \end{cases} \begin{cases} y_1 D = w_1 \\ \vdots \\ y_n D = w_n \\ y_{n+1} D = w_n \\ \vdots \\ y_{2n} D = w_1 \\ y_{2n+1} D = 0 \end{cases}$$

Observe that all the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2n+1} , so clearly the dimensions of the solution spaces for $(**)^1$ and

$$(***)^1 \quad \begin{cases} y_{n+1} A = 0 \\ y_{n+2} A = 0 \\ \vdots \\ y_{2n+1} A = 0 \end{cases} \begin{cases} y_1 B = 0 \\ y_2 B = 0 \\ \vdots \\ y_{n+1} B = 0 \end{cases} \begin{cases} y_1 C = 0 \\ y_2 C - y_{2n+1} C = 0 \\ y_3 C - y_{2n} C = 0 \\ \vdots \\ y_{n+1} C - y_{n+2} C = 0 \end{cases} \begin{cases} y_1 D - y_{2n} D = 0 \\ y_2 D - y_{2n-1} D = 0 \\ \vdots \\ y_n D - y_{n+1} D = 0 \\ y_{2n+1} D = 0 \end{cases}$$

are equal.

For the benefit of the reader we discuss in details the cases $n = 1$ and 2 . For $n = 1$, the system $(***)^1$ has the form

$$\begin{cases} y_2 A = 0 \\ y_3 A = 0 \end{cases} \begin{cases} y_1 B = 0 \\ y_2 B = 0 \end{cases} \begin{cases} y_1 C = 0 \\ y_2 C - y_3 C = 0 \end{cases} \begin{cases} y_1 D - y_2 D = 0 \\ y_3 D = 0 \end{cases}$$

and we can equivalently write it in the following matrix form

$$[y_2 \ y_1 \ y_3] \begin{bmatrix} A & 0 & B & 0 & C & D & 0 & 0 \\ 0 & 0 & 0 & B & 0 & -D & C & 0 \\ 0 & A & 0 & 0 & -C & 0 & 0 & D \end{bmatrix} = 0$$

Note that the matrix appearing in the above equation equals

$$\mathcal{N}_1 = \mathcal{M}_3(Z_{P(1,0)}, Z'_{P(1,0)}, C \oplus D, 0),$$

so the dimension of the solution space of this equation, which is equal to $[M, P(1, 0)]$, is $\text{cor}(\mathcal{N}_1)$, and this is precisely the assertion for $n = 1$.

For $n = 2$, the system $(***)^1$ has the form

$$\begin{cases} y_3 A = 0 \\ y_4 A = 0 \\ y_5 A = 0 \end{cases} \quad \begin{cases} y_1 B = 0 \\ y_2 B = 0 \\ y_3 B = 0 \end{cases} \quad \begin{cases} y_1 C = 0 \\ y_2 C - y_5 C = 0 \\ y_3 C - y_4 C = 0 \end{cases} \quad \begin{cases} y_1 D - y_4 D = 0 \\ y_2 D - y_3 D = 0 \\ y_5 D = 0 \end{cases}$$

and we can equivalently write it in the following matrix form

$$[y_3 \ y_2 \ y_4 \ y_1 \ y_5] \left[\begin{array}{cccccc|cccc|cc} A & 0 & B & 0 & C & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & -D & C & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & -C & 0 & 0 & D & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -D & 0 & B & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -C & 0 & A & 0 & 0 & D \end{array} \right] = 0$$

Note that the matrix appearing in the above equation equals

$$\mathcal{N}_2 = \mathcal{M}_3(Z_{P(2,0)}, Z'_{P(2,0)}, C \oplus D, 1),$$

so the dimension of the solution space of this equation, which is equal to $[M, P(2, 0)]$, is $\text{cor}(\mathcal{N}_2)$, and this is precisely the assertion for $n = 2$.

Similarly one checks that for the general case $n \geq 1$, the system $(***)^1$ is equivalent to the matrix equation

$$[y_{n+1} \ y_n \ y_{n+2} \ y_{n-1} \ y_{n+3} \ y_{n-2} \ \dots \ y_{2n} \ y_1 \ y_{2n+1}] \cdot \mathcal{N}_n = 0,$$

where $\mathcal{N}_n = \mathcal{M}_3(Z_{P(n,0)}, Z'_{P(n,0)}, C \oplus D, n-1)$. Therefore the dimension $[M, P(n, 0)]$ equals $\text{cor}(\mathcal{N}_n)$ and we are done.

Case 2: $X = P(2n+1, 1)$. Fix $n \geq 0$. Then every homomorphism from M to $P(2n+1, 1)$ is given by the collection of matrices $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{(2n+2) \times n_0} \times \mathbb{M}_{n \times n_1} \times \mathbb{M}_{(n+1) \times n_2} \times \mathbb{M}_{(n+1) \times n_3} \times \mathbb{M}_{(n+1) \times n_4}$ satisfying the commutativity relations

$$(*)^2 \quad F_0 A = \begin{bmatrix} 0_{1,n} \\ I_n \\ I_n \\ 0_{1,n} \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} I_{n+1} \\ 0_{n+1} \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} 0_{n+1} \\ I_{n+1} \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} I_{n+1} \\ I_{n+1} \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by $y_1, \dots, y_{2n+2}, s_1, \dots, s_n, t_1, \dots, t_{n+1}, u_1, \dots, u_{n+1}$ and respectively w_1, \dots, w_{n+1} , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . Similarly as in the case 1, the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2n+2} , so clearly the dimensions of the solution spaces for $(*)^2$ and

$$(**)^2 \quad \begin{cases} y_1 A = 0 \\ y_2 A - y_{n+2} A = 0 \\ y_3 A - y_{n+3} A = 0 \\ \vdots \\ y_{n+1} A - y_{2n+1} A = 0 \\ y_{2n+2} A = 0 \end{cases} \quad \begin{cases} y_{n+2} B = 0 \\ y_{n+3} B = 0 \\ \vdots \\ y_{2n+2} B = 0 \end{cases} \quad \begin{cases} y_1 C = 0 \\ y_2 C = 0 \\ \vdots \\ y_{n+1} C = 0 \end{cases} \quad \begin{cases} y_1 D - y_{n+2} D = 0 \\ y_2 D - y_{n+3} D = 0 \\ \vdots \\ y_{n+1} D - y_{2n+2} D = 0 \end{cases}$$

are equal. One checks that the system $(**)^2$ is equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_{n+2} & y_2 & y_{n+3} & y_3 & y_{n+4} & \cdots & y_n & y_{2n+1} & y_{n+1} & y_{2n+2} \end{bmatrix} \cdot \mathcal{N}_n = 0,$$

where $\mathcal{N}_n = \mathcal{M}_3(Z_{P(2n+1,1)}, Z_{P(2n+1,1)}, -A, n)$. Therefore the dimension $[M, P(2n+1, 1)]$ equals $\text{cor}(\mathcal{N}_n)$.

Case 3: $X = P(2n+1, i)$, $i \in \{2, 3, 4\}$. The assertions for this case follow immediately from the case 2, since the module $P(2n+1, i)$, for $i \in \{2, 3, 4\}$, can be obtained from $P(2n+1, i-1)$ after the cyclic permutation $(1, 2, 3, 4)$ of vertices of the quiver Q (see Proposition 2).

Case 4: $X = P(2n, 1)$. Fix $n \geq 0$. Then every homomorphism from M to $P(2n, 1)$ is given by the collection of matrices $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{(2n+1) \times n_0} \times \mathbb{M}_{(n+1) \times n_1} \times \mathbb{M}_{n \times n_2} \times \mathbb{M}_{n \times n_3} \times \mathbb{M}_{n \times n_4}$ satisfying the commutativity relations

$$(*)^4 \quad F_0 A = \begin{bmatrix} I_{n+1} \\ 0_{n, n+1} \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} 0_{n+1, n} \\ I_n \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} 0_{1, n} \\ I_n \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} I_n \\ 0_{1, n} \\ I_n \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by y_1, \dots, y_{2n+1} , s_1, \dots, s_{n+1} , t_1, \dots, t_n , u_1, \dots, u_n and respectively w_1, \dots, w_n , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . Similarly as before, the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2n+1} , so clearly the dimensions of the solution spaces for $(*)^4$ and

$$(**)^4 \quad \begin{cases} y_{n+2}A = 0 \\ y_{n+3}A = 0 \\ \vdots \\ y_{2n+1}A = 0 \end{cases} \quad \begin{cases} y_1B = 0 \\ y_2B = 0 \\ \vdots \\ y_{n+1}B = 0 \end{cases} \quad \begin{cases} y_1C = 0 \\ y_2C - y_{n+2}C = 0 \\ y_3C - y_{n+3}C = 0 \\ \vdots \\ y_{n+1}C - y_{2n+1}C = 0 \end{cases} \quad \begin{cases} y_1D - y_{n+2}D = 0 \\ y_2D - y_{n+3}D = 0 \\ \vdots \\ y_nD - y_{2n+1}D = 0 \\ y_{n+1}D = 0 \end{cases}$$

are equal.

For $n = 0$, the system $(**)^4$ has the form $y_1B = 0$, $y_1C = 0$ and $y_1D = 0$, and we can write it in the matrix form as $y_1\mathcal{N}_0 = 0$, where $\mathcal{N}_0 = [B \ C \ D]$. So the dimension $[M, P(0, 1)]$ equals $\text{cor}(\mathcal{N}_0)$ and this is the assertion for $n = 0$, since $\mathcal{N}_0 = \mathcal{M}_1(Z_{P(0,1)}, Z'_{P(0,1)}, -D, 0)$.

Similarly one checks that in the general case $n \geq 0$, the system $(**)^4$ is equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_{n+2} & y_2 & y_{n+3} & y_3 & y_{n+4} & \cdots & y_n & y_{2n+1} & y_{n+1} \end{bmatrix} \cdot \mathcal{N}_n = 0,$$

where $\mathcal{N}_n = \mathcal{M}_1(Z_{P(2n,1)}, Z'_{P(2n,1)}, -D, n)$. Therefore the dimension $[M, P(2n, 1)]$ equals $\text{cor}(\mathcal{N}_n)$.

Case 5: $X = P(2n, i)$, $i \in \{2, 3, 4\}$. The assertions for this case follow immediately from the case 4, since the module $P(2n, i)$, for $i \in \{2, 3, 4\}$, can be obtained from $P(2n, i-1)$ after the cyclic permutation $(1, 2, 3, 4)$ of vertices of the quiver Q (see Proposition 2).

Case 6: $X = I(n, 0)$. $I(0, 0)$ is the injective Λ -module $I(0) = ([1], [1], [1], [1])$. So every homomorphism from M to $I(0, 0)$ is given by the collection $(y, s, t, u, w) \in \mathbb{M}_{1 \times n_0} \times \mathbb{M}_{1 \times n_1} \times \mathbb{M}_{1 \times n_2} \times \mathbb{M}_{1 \times n_3} \times \mathbb{M}_{1 \times n_4}$, such that $yA = s$, $yB = t$, $yC = u$, $yD = w$ that is, the variables s, t, u, w are determined by y and y can have an arbitrary value. Thus $[M, I(0, 0)] = \dim(\mathbb{M}_{1 \times n_0}) = n_0$.

For $n \geq 1$, every homomorphism $F : M \rightarrow I(n, 0)$ is given by the collection $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{(2n+1) \times n_0} \times \mathbb{M}_{(n+1) \times n_1} \times \mathbb{M}_{(n+1) \times n_2} \times \mathbb{M}_{(n+1) \times n_3} \times \mathbb{M}_{(n+1) \times n_4}$ satisfying the commutativity relations

$$(*)^6 \quad F_0 A = \begin{bmatrix} 0_{n, n+1} \\ I_{n+1} \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} I_{n+1} \\ 0_{n, n+1} \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} \bar{I}_{n+1} \\ \circ \pi_{n, n+1} \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} \pi_{n, n+1}^\circ \\ \bar{I}_{n+1} \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by y_1, \dots, y_{2n+1} , s_1, \dots, s_{n+1} , t_1, \dots, t_{n+1} , u_1, \dots, u_{n+1} and respectively w_1, \dots, w_{n+1} , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . Similarly as before, the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2n+1} , so clearly the dimensions of the solution spaces for $(*)^6$ and

$$(**)^6 \quad \begin{cases} y_1 A = 0 \\ y_2 A = 0 \\ \vdots \\ y_n A = 0 \end{cases} \quad \begin{cases} y_{n+2} B = 0 \\ y_{n+3} B = 0 \\ \vdots \\ y_{2n+1} B = 0 \end{cases} \quad \begin{cases} y_2 C - y_{2n+1} C = 0 \\ y_3 C - y_{2n} C = 0 \\ \vdots \\ y_n C - y_{n+3} C = 0 \\ y_{n+1} C - y_{n+2} C = 0 \end{cases} \quad \begin{cases} y_1 D - y_{2n} D = 0 \\ y_2 D - y_{2n-1} D = 0 \\ \vdots \\ y_{n-1} D - y_{n+2} D = 0 \\ y_n D - y_{n+1} D = 0 \end{cases}$$

are equal.

One checks that the system $(**)^6$ is equivalent to the matrix equation

$$[y_{n+1} \ y_{n+2} \ y_n \ y_{n+3} \ y_{n-1} \ y_{n+4} \ \dots \ y_2 \ y_{2n+1} \ y_1] \cdot \mathcal{N}_n = 0,$$

where $\mathcal{N}_n = \mathcal{M}_2(Z_{I(n,0)}, Z'_{I(n,0)}, D \oplus C, n-1)$. Therefore the dimension $[M, I(n,0)]$ equals $\text{cor}(\mathcal{N}_n)$.

Case 7: $X = I(2n+1, 1)$. $I(1, 1)$ has the form $(\varepsilon_{1,0}, [1], [1], [1])$. So every homomorphism $F : M \rightarrow I(1, 1)$ is given by the collection $(y, 0, t, u, w) \in \mathbb{M}_{1 \times n_0} \times \mathbb{M}_{0 \times n_1} \times \mathbb{M}_{1 \times n_2} \times \mathbb{M}_{1 \times n_3} \times \mathbb{M}_{1 \times n_4}$, such that $yA = 0$, $yB = t$, $yC = u$, $yD = w$. Therefore the variables t, u, w are determined by y , so the dimension $[M, I(1, 1)]$ equals $\text{cor}(A)$ and this is precisely the assertion for $n = 0$ since $A = \mathcal{M}_2(Z_{I(1,1)}, A, -C, 0)$.

For $n \geq 1$, every homomorphism $F : M \rightarrow I(2n+1, 1)$ is given by the collection $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{(2n+1) \times n_0} \times \mathbb{M}_{n \times n_1} \times \mathbb{M}_{(n+1) \times n_2} \times \mathbb{M}_{(n+1) \times n_3} \times \mathbb{M}_{(n+1) \times n_4}$ satisfying the commutativity relations

$$(*)^7 \quad F_0 A = \begin{bmatrix} 0_{n+1,n} \\ I_n \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} I_{n+1} \\ 0_{n,n+1} \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} I_{n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} I_{n+1} \\ \pi_{n,n+1}^\circ \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by y_1, \dots, y_{2n+1} , s_1, \dots, s_n , t_1, \dots, t_{n+1} , u_1, \dots, u_{n+1} and respectively w_1, \dots, w_{n+1} , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . Similarly as before, the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2n+1} , so clearly the dimensions of the solution spaces for $(*)^7$ and

$$(**)^7 \quad \begin{cases} y_1 A = 0 \\ \vdots \\ y_n A = 0 \\ y_{n+1} A = 0 \end{cases} \quad \begin{cases} y_{n+2} B = 0 \\ y_{n+3} B = 0 \\ \vdots \\ y_{2n+1} B = 0 \end{cases} \quad \begin{cases} y_2 C - y_{n+2} C = 0 \\ y_3 C - y_{n+3} C = 0 \\ \vdots \\ y_{n+1} C - y_{2n+1} C = 0 \end{cases} \quad \begin{cases} y_1 D - y_{n+2} D = 0 \\ y_2 D - y_{n+3} D = 0 \\ \vdots \\ y_{n-1} D - y_{2n} D = 0 \\ y_n D - y_{2n+1} D = 0 \end{cases}$$

are equal.

One checks that the system $(**)^7$ is equivalent to the matrix equation

$$[y_{n+1} \ y_{2n+1} \ y_n \ y_{2n} \ y_{n-1} \ y_{2n-1} \ \dots \ y_2 \ y_{n+2} \ y_1] \cdot \mathcal{N}_n = 0,$$

where $\mathcal{N}_n = \mathcal{M}_2(Z_{I(2n+1,1)}, A, -C, n)$. Therefore the dimension $[M, I(2n+1, 1)]$ equals $\text{cor}(\mathcal{N}_n)$.

Case 8: $X = I(2n+1, i)$, $i \in \{2, 3, 4\}$. The assertions for this case follow immediately from the case 7, since the module $I(2n+1, i)$, for $i \in \{2, 3, 4\}$, can be obtained from $I(2n+1, i-1)$ after the cyclic permutation $(1, 2, 3, 4)$ of vertices of the quiver Q (see Proposition 2).

Case 9: $X = I(2n, 1)$. $I(0, 1)$ is the injective Λ -module $I(1) = (\varepsilon_{0,1}, \varepsilon_{0,0}, \varepsilon_{0,0}, \varepsilon_{0,0})$. So every homomorphism $F : M \rightarrow I(0, 1)$ is given by the collection $(0, s, 0, 0, 0)$

where $s \in \mathbb{M}_{1 \times n_1}$ can have an arbitrary value (see $(*)$). So $[M, I(0, 1)] = \dim(\mathbb{M}_{1 \times n_1}) = n_1$.

For $n \geq 1$, every homomorphism $F : M \rightarrow I(2n, 1)$ is given by the collection $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{2n \times n_0} \times \mathbb{M}_{(n+1) \times n_1} \times \mathbb{M}_{n \times n_2} \times \mathbb{M}_{n \times n_3} \times \mathbb{M}_{n \times n_4}$ satisfying the commutativity relations

$$(*)^9 \quad F_0 A = \begin{bmatrix} \circ & \pi_{n,n+1} \\ \pi_{n,n+1} & \circ \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} 0_n \\ I_n \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} I_n \\ 0_n \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} I_n \\ I_n \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by $y_1, \dots, y_{2n}, s_1, \dots, s_{n+1}, t_1, \dots, t_n, u_1, \dots, u_n$ and respectively w_1, \dots, w_n , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . Similarly as before, the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2n} , so clearly the dimensions of the solution spaces for $(*)^9$ and

$$(**)^9 \quad \begin{cases} y_2 A - y_{n+1} A = 0 \\ y_3 A - y_{n+2} A = 0 \\ \vdots \\ y_{n-1} A - y_{2n-2} A = 0 \\ y_n A - y_{2n-1} A = 0 \end{cases} \quad \begin{cases} y_1 B = 0 \\ y_2 B = 0 \\ \vdots \\ y_n B = 0 \end{cases} \quad \begin{cases} y_{n+1} C = 0 \\ y_{n+2} C = 0 \\ \vdots \\ y_{2n} C = 0 \end{cases} \quad \begin{cases} y_1 D - y_{n+1} D = 0 \\ y_2 D - y_{n+2} D = 0 \\ \vdots \\ y_{n-1} D - y_{2n-1} D = 0 \\ y_n D - y_{2n} D = 0 \end{cases}$$

are equal.

One checks that the system $(**)^9$ is equivalent to the matrix equation

$$[y_1 \ y_{n+1} \ y_2 \ y_{n+2} \ y_3 \ y_{n+3} \ \dots \ y_n \ y_{2n}] \cdot \mathcal{N}_n = 0,$$

where $\mathcal{N}_n = \mathcal{M}_2(Z_{I(2n,1)}, Z'_{I(2n,1)}, -A, n-1)$. Therefore the dimension $[M, I(2n, 1)]$ equals $\text{cor}(\mathcal{N}_n)$.

Case 10: $X = I(2n, i)$, $i \in \{2, 3, 4\}$. The assertions for this case follow immediately from the case 9, since the module $I(2n, i)$, for $i \in \{2, 3, 4\}$, can be obtained from $I(2n, i-1)$ after the cyclic permutation $(1, 2, 3, 4)$ of vertices of the quiver Q (see Proposition 2).

Case 11: $X = R(l, \lambda)$. For $l \geq 1$, every homomorphism from M to $R(l, \lambda)$ is given by the collection of matrices $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{2l \times n_0} \times \mathbb{M}_{l \times n_1} \times \mathbb{M}_{l \times n_2} \times \mathbb{M}_{l \times n_3} \times \mathbb{M}_{l \times n_4}$ satisfying the commutativity relations

$$(*)^{11} \quad F_0 A = \begin{bmatrix} I_l \\ 0_l \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} 0_l \\ I_l \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} I_l \\ I_l \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} J_l^{(\lambda)} \\ I_l \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by $y_1, \dots, y_l, y_{l+1}, \dots, y_{2l}, s_1, \dots, s_l, t_1, \dots, t_l, u_1, \dots, u_l$ and respectively w_1, \dots, w_l , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . So the system $(*)^{11}$ can be presented in the form

$$(**)^{11} \quad \begin{cases} y_1 A = s_1 \\ \vdots \\ y_l A = s_l \\ y_{l+1} A = 0 \\ \vdots \\ y_{2l} A = 0 \end{cases} \quad \begin{cases} y_1 B = 0 \\ \vdots \\ y_l B = 0 \\ y_{l+1} B = t_1 \\ \vdots \\ y_{2l} B = t_l \end{cases} \quad \begin{cases} y_1 C = u_1 \\ \vdots \\ y_l C = u_l \\ y_{l+1} C = u_1 \\ \vdots \\ y_{2l} C = u_l \end{cases} \quad \begin{cases} y_1 D = \lambda w_1 + w_2 \\ \vdots \\ y_{l-1} D = \lambda w_{l-1} + w_l \\ y_l D = \lambda w_l \\ y_{l+1} D = w_1 \\ \vdots \\ y_{2l} D = w_l \end{cases}$$

Since again all the variables s_i, t_i, u_i, w_i are determined by x_1, \dots, x_{2l} , so clearly the dimensions of the solution spaces for $(**)^{11}$ and

$$(***)^{11} \quad \begin{cases} y_{l+1} A = 0 \\ y_{l+2} A = 0 \\ \vdots \\ y_{2l} A = 0 \end{cases} \quad \begin{cases} y_1 B = 0 \\ y_2 B = 0 \\ \vdots \\ y_l B = 0 \end{cases} \quad \begin{cases} y_1 C - y_{l+1} C = 0 \\ y_2 C - y_{l+2} C = 0 \\ \vdots \\ y_l C - y_{2l} C = 0 \end{cases} \quad \begin{cases} y_1 D - \lambda y_{l+1} D - y_{l+2} D = 0 \\ \vdots \\ y_{l-1} D - \lambda y_{2l-1} D - y_{2l} D = 0 \\ y_l D - \lambda y_{2l} D = 0 \end{cases}$$

are equal. One easily checks that the system $(***)^{11}$ is equivalent to the matrix equation

$$[y_l \ y_{2l} \ y_{l-1} \ y_{2l-1} \ y_{l-2} \ y_{2l-2} \ \dots \ y_1 \ y_{l+1}] \cdot \mathcal{N}_l = 0,$$

where $\mathcal{N}_l = \mathcal{M}_2(Z_{R(l,\lambda)}, Z_{R(l,\lambda)}, -D, l-1)$ is the matrix from the assertion. Therefore the dimension $[M, R(l, \lambda)]$ is the dimension of the solution space of the above equation which equals $\text{cor}(\mathcal{N}_l)$.

Case 12: $X = R(s, 2l, \lambda)$, $s \in \mathbb{Z}_2$, $\lambda \in \{0, 1, \infty\}$. Observe that for every $l \geq 1$, $R(0, 2l, 0)$ is just the module $R(l, \lambda)$ for $\lambda := 0$ (see Proposition 2). So we obtain the assertion for $R(0, 2l, 0)$ immediately from the assertion for $R(l, \lambda)$ when we substitute $\lambda := 0$ (cf. case 11).

The assertions for the remaining modules in this case follow obviously from the above case of $R(0, 2l, 0)$ since the modules $R(s, 2l, \lambda)$ can be obtained from $R(0, 2l, 0)$ after suitable permutation of the vertices of quiver Q (see Proposition 2).

Case 13: $X = R(0, 2l-1, 0)$. For $l \geq 1$, every homomorphism $F : M \rightarrow R(0, 2l-1, 0)$ is given by the collection $(F_0, F_1, F_2, F_3, F_4) \in \mathbb{M}_{(2l-1) \times n_0} \times \mathbb{M}_{(l-1) \times n_1} \times \mathbb{M}_{l \times n_2} \times \mathbb{M}_{(l-1) \times n_3} \times \mathbb{M}_{l \times n_4}$ satisfying the commutativity relations

$$(*)^{13} \quad F_0 A = \begin{bmatrix} I_{l-1} \\ 0_{1,l-1} \\ I_{l-1} \end{bmatrix} F_1, \quad F_0 B = \begin{bmatrix} \circ \pi_{l-1,l} \\ I_l \end{bmatrix} F_2, \quad F_0 C = \begin{bmatrix} I_{l-1} \\ 0_{l,l-1} \end{bmatrix} F_3, \quad F_0 D = \begin{bmatrix} 0_{l-1,l} \\ I_l \end{bmatrix} F_4$$

(see $(*)$ and Proposition 2). Let us denote by y_1, \dots, y_{2l-1} , s_1, \dots, s_{l-1} , t_1, \dots, t_l , u_1, \dots, u_{l-1} and respectively w_1, \dots, w_l , the consecutive rows of matrices F_0, F_1, F_2, F_3, F_4 . Similarly as before, the variables s_i, t_i, u_i, w_i are determined by y_1, \dots, y_{2l-1} , so clearly the dimensions of the solution spaces for $(*)^{13}$ and

$$(**)^{13} \quad \begin{cases} y_l A & = 0 \\ y_1 A - y_{l+1} A & = 0 \\ y_2 A - y_{l+2} A & = 0 \\ \vdots & \\ y_{l-1} A - y_{2l-1} A & = 0 \end{cases} \quad \begin{cases} y_1 B - y_l B & = 0 \\ y_2 B - y_{l+1} B & = 0 \\ \vdots & \\ y_{l-1} B - y_{2l-2} B & = 0 \end{cases} \quad \begin{cases} y_l C & = 0 \\ y_{l+1} C & = 0 \\ \vdots & \\ y_{2l-1} C & = 0 \end{cases} \quad \begin{cases} y_1 D & = 0 \\ y_2 D & = 0 \\ \vdots & \\ y_{l-1} D & = 0 \end{cases}$$

are equal.

One checks that the system $(**)^{13}$ is equivalent to the matrix equation

$$[y_l \ y_1 \ y_{l+1} \ y_2 \ y_{l+2} \ y_3 \ \dots \ y_{2l-2} \ y_{l-1} \ y_{2l-1}] \cdot \mathcal{N}_l = 0,$$

where $\mathcal{N}_l = \mathcal{M}_2(Z_{R(0,2l-1,0)}, Z'_{R(0,2l-1,0)}, -B, l-1)$. So the dimension $[M, R(0, 2l, 0)]$ equals $\text{cor}(\mathcal{N}_l)$.

Case 14: $X = R(s, 2l-1, \lambda)$, $s \in \mathbb{Z}_2$, $\lambda \in \{0, 1, \infty\}$ and $(s, \lambda) \neq (0, 0)$. The assertions for this case follow immediately from the case 13 since the modules $R(s, 2l-1, \lambda)$ can be obtained from $R(0, 2l-1, 0)$ after suitable permutation of the vertices of quiver Q (see Proposition 2).

We considered above all the cases of the assertion so the proof is finished. \square

Remark. Observe that all the matrices $\mathcal{M}_*(-, -, -, -)$ appearing in the assertion of Proposition 1 have very specific, “almost block diagonal” shapes, which is essentially used in the paper [3] to decrease the computational complexity of considered algorithms. We obtained those shapes thanks to determining the right orders of the equations and the variables y_i in the matrix equations in the proof above. Determining those orders was not always an easy task!

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